REVIEW NOTES FOR REAL AND COMPLEX ANALYSIS CHAPTER1-5

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TO YIFAN
This is the notes that I made in preparation for an exam on Real Analysis. Please inform me if there is any mistake.

1. Preliminary

1.1. Sets and Functions.

Theorem 1. *If* $\alpha, \beta \in \Re$, *then* $(\alpha, \beta] = \bigcup_{n=1}^{\infty}(\alpha + \frac{1}{n}, \beta] = \bigcup_{n=1}^{\infty}[\alpha + \frac{1}{n}, \beta] = \bigcap_{n=1}^{\infty}(\alpha, \beta + \frac{1}{n}) = \bigcap_{n=1}^{\infty}(\alpha, \beta + \frac{1}{n}]$.

Theorem 2. *1.* $f(\bigcup A_\alpha) = \bigcup f(A_\alpha)$
2. $f(\bigcap A_\alpha) \subset \cap f(A_\alpha)$
3. $f^{-1}(\bigcup A_\alpha) = \bigcup f^{-1}(A_\alpha)$
4. $f^{-1}(\bigcap A_\alpha) = \bigcap f^{-1}(A_\alpha)$
5. $f^{-1}(A^c) = (f^{-1}(A))^c$
6. $f(f^{-1}(A)) \subset A$
7. If $F = (f_1.f_2.f_3, \ldots, f_n)$, *then* $f^{-1}(A_1 \times A_2 \times A_3 \times \cdots \times A_n) = \bigcap_1^n f_i^{-1}(A_i)$

Theorem 3. *Let* $f : X \to Y$, *where* X *and* Y *are topological spaces, then* f *is continuos if and only if it is locally continuos at all* $x \in X$.

Theorem 4. *Let* f *be a complex function on a topological space* X, *then the set of points at which* f *is continuos is a* G_δ.

Theorem 5. *A real function is continuos if and only if it is both upper and lower semicontinuos.*

Theorem 6. *1. Characteristic functions of open sets are lower semicontinuos.*
2. Characteristic functions of closed sets are upper semicontinuos.

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3. The supremum of any collection of lower semicontinuos functions is lower semicontinuos. The infimum of any collection of upper semicontinuos functions is upper semicontinuos.

Theorem 7. *Let $f : X \to Y$ be continuos. then $f(K)$ is compact in Y whenever K is compact in X.*

Theorem 8. *If X is compact and $f : X \to (-\infty, +\infty)$ is upper semicontinuos, then f attains its maximum at some point in X.*

Theorem 9. *Suppose that*
(a) X and Y are metric spaces and X is complete,
(b) $f : X \to Y$ is continuos,
(c) X has a dense subset X_0 on which f is an isometry, and
(d) $f(X_0)$ is dense in Y.
Then f is an isometry of X onto Y.

Theorem 10. *A real function f is measurable if and only if $f^{-1}((\alpha, +\infty])$ is measurable for all $\alpha \in \Re$.*

Theorem 11. *If $f_n : X \to [-\infty, +\infty]$ is measurable for $n \in \mathcal{N}$ and $g = sup f_n$, $h = \overline{\lim} f_n$, then g and h are measurable.*

Theorem 12. *Let Y and Z be topological spaces and $g : Y \to Z$ be continuos.*
(a)If X is a topological space and $f : X \to Y$ is continuos, then $h = g \circ f$ is continuos.
(b)If X is measurable space and $f : X \to Y$ is measurable, then $h = g \circ f$ is measurable.

Theorem 13. *Let u and v be real measurable functions on a measurable space X, let Φ be a continuos mapping of the plane into a topological space Y. Then $h(x) = \Phi(u(x), v(x))$ is measurable.*

Theorem 14. *(a) A complex function is measurable if and only if both of its real part and complex part are measurable functions.*
(b)The sum/ product of complex measurable functions is measurable.
(c) If f is a complex measurable function on X, there is a complex function α with $|\alpha| = 1$ and $f = \alpha|f|$.
(d) If f and g are real measurable functions, then $\{f(x) < g(x)\}$ and $\{f(x) = g(x)\}$ are measurable.
(e) The set of points at which a sequence of measurable real functions converges is measurable.

Theorem 15. *The supremum of any collection of convex functions on (a, b) is convex. So is their infimum and hence limits.*

Theorem 16. *The Hausdorff Maximality Theorem*
Every nonempty partially ordered set contains a maximal totally ordered subset.

1.2. Complex Numbers.

Theorem 17. *If* $z_1, z_2, \ldots, z_N$ *are complex numbers then there is a subset* S *of* $\{1, 2, 3 \ldots, N\}$ *such that* $|\sum_{k\in S} z_k| \geq \frac{1}{\pi}\sum_1^N |z_k|$.

1.3. Topology.

Theorem 18. *Suppose* K *is compact and* F *is closed in a topological space* X. *If* $F \subset K$, *then* F *is compact.*

Corollary 1. *If* $A \subset B$ *and* B *has compact closure, so does* A.

Theorem 19. *Suppose* X *is a Hausdorff space,* $K \subset X$, K *is compact, and* $p \in K^c$. *Then there are open sets* U *and* W *such that* $p \in U$ *and* $K \subset W$, *and* $U \cap W = \emptyset$.

Corollary 2. *1. Compact subsets of Hausdorff spaces are closed.*
2. If F *is closed and* K *is compact in a Hausdorff space, then* $F \cap K$ *is compact.*

Theorem 20. *If* $\{K_\alpha\}$ *is a collection of compact subsets of a Hausdorff space and if* $\bigcap_\alpha K_\alpha = \emptyset$, *then some finite subcollection of* $\{K_\alpha\}$ *also has empty intersection.*

Theorem 21. *In metric spaces, compactness is equivalent to sequential compactness.*

Theorem 22. *Suppose* U *is open in a locally compact Hausdorff space* X, $K \subset U$, *and* K *is compact. Then there is an open set* V *with compact closure such that* $K \subset V \subset \overline{V} \subset U$.

Theorem 23. *Urysohn's Lemma*
Suppose X *is a locally compact Hausdorff space,* V *is open in* X, $K \subset V$ *and* K *is compact. Then there is an* $f \in C_c(X)$. *such that* $K \prec f \prec V$.

Theorem 24. *Suppose* $V_1, \ldots, V_n$ *are open subsets of a locally compact Hausdorff space* X, K *is compact, and* $K \subset \bigcup_1^n V_i$, *then there exits functions* $h_i \prec V_i$ *such that* $h_1 + \cdots + h_n = 1$ *for all* $x \in K$.

Theorem 25. *Baire's Theorem*
If X *is a complete metric space, the intersection of every countable collection of dense open subsets of* X *is dense in* X.

Theorem 26. *In a complete metric space* X *which has no isolated point, no countable dense set is a* G_δ.

2. Measurable Sets and Measures

2.1. Measurable Sets and Positive Measures.

Theorem 27. *$\mathcal{F}$ is a collection of subsets of X,then there is a smallest σ-algebra $\mathfrak{M}^*$ such that $\mathcal{F} \subset \mathfrak{M}^*$.*

Theorem 28. *There does not exist an infinite σ-algebra which has only countable members.*

Theorem 29. *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $\mathfrak{M}^*$ be the collection of all $E \subset X$ for which there exist sets A and $B \in \mathfrak{M}$ such that $A \subset E \subset B$ and $\mu(B - A) = 0$, and define $\mu(E) = \mu(A)$. Then $\mathfrak{M}^*$ is aσ-algebra and μ is a measure on $\mathfrak{M}^*$.*

Theorem 30. *Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then*

(a) $\mu(\emptyset) = 0$.

(b) $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \mu(A_2) + \cdots + \mu(A_n)$ if $A_1, \ldots, A_n$ are pairwise disjoint members of $\mathfrak{M}$.

(c) $A \subset B$ implies $\mu(A) \leq \mu(B)$.

(d) If $A = \bigcup_1^\infty A_n$, $A_n \in \mathfrak{M}$ and $A_1 \subset A_2 \subset \ldots$, then $\mu(A_n) \to \mu(A)$ as $n \to \infty$.

(e)If $A = \bigcap_1^\infty A_n$, $A_n \in \mathfrak{M}^*$, and $A_1 \supset A_2 \supset A_3 \ldots$ and $\mu(A_1) < +\infty$, then $\mu(A_n) \to \mu(A)$ as $n \to \infty$.

Theorem 31. *Let $\{E_k\}$ be a sequence of measurable sets in X such that $\sum_1^\infty \mu(E_k) < \infty$. Then almost all $x \in X$ lie in at most finitely many of E_k's.*

Theorem 32. *Suppose $\mathfrak{M}$ is a σ-algebra in X, and Y is a topological space. Let f map X into Y.*

(a) If Ω is the collection of all sets $E \subset Y$ such that $f^{-1}(E) \in \mathfrak{M}$, then Ω is a σ-algebra in Y.

(b) If f is measurable and E is a Borel set in Y, then $f^{-1}(E) \in \mathfrak{M}$.

(c) If $Y = [-\infty, +\infty]$ and $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$ for every real α, then f is measurable.

(d) If f is measurable and Z is topological and $g : Y \to Z$ is a Borel mapping, and if $h = g \circ f$, then h is measurable.

Theorem 33. *Let s and t be nonnegative measurable simple functions on X. For $E \in \mathfrak{M}$, difine $\phi(E) = \int_E s d\mu$. Then ϕ is a measure on $\mathfrak{M}$. Also $\int_X (s + t) d\mu = \int_X s d\mu + \int_X t d\mu$.*

Theorem 34. *Suppose $f : X \to [0, \infty]$ is measurable, and $\phi(E) = \int_E f d\mu$ for all $E \in \mathfrak{M}$, then ϕ is a measure on $\mathfrak{M}$, and $\int_X g d\phi = \int_X g f d\mu$ for every measurable g on X with range in $[0, \infty]$.*

2.2. **Measurable Sets and Positive Measures in Locally Compact Hausdorff Spaces.**

Theorem 35. *Suppose X is a locally compact, σ-compact Hausdorff space. If $\mathfrak{M}$ and μ are as described in the statement of Riesz Representation Theorem, then $\mathfrak{M}$ and μ have the following properties:*

(a) If $E \in \mathfrak{M}$ and $\epsilon > 0$, there is a closed set F and an open set V such that $F \subset E \subset V$ and $\mu(V - F) < \epsilon$.

(b) μ is a regular Borel measure on X.

(c) If $E \in \mathfrak{M}$, there are sets A and B such that A is an F_σ and B is a G_δ, and $A \subset E \subset B$, and $\mu(B - A) = 0$.

Theorem 36. *Let X be a locally compact Hausdorff space in which every open set is σ-compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K. Then λ is regular.*

Theorem 37. *Let μ be a regular Borel measure on a compact Hausdorff space X; assume $\mu(X) = 1$. Then there is a compact set K such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K.*

Theorem 38. *Every compact subset of $\Re$ is the support of a Borel measure.*

Theorem 39. *Existence of Lebesgue Measure*

There exists a positive complete measure m defined on a σ-algebra $\mathfrak{M}$ in $\Re^k$, with the following properties:

(a)$m(W) = vol(W)$ for every k-cell W.

(b)$\mathfrak{M}$ contains all Borel sets in $\Re^k$; $E \in \mathfrak{M}$ if and only if there are sets A and $B \subset \Re^k$ such that $A \subset E \subset B$, and A is an F_σ, B is a G_δ, and $m(B-A) = 0$. Also, m is regular.

(c)m is translation-invariant, i.e., $m(E + x) = m(E)$ for all $E \in \mathfrak{M}$ and $x \in \Re^k$.

(d)If μ is any positive translation-invariant Borel measure on $\Re^k$ such that $\mu(K) < \infty$ for every compact set K, then there is a constant c such that $\mu(E) = cm(E)$ for all Borel sets $E \subset \Re^k$.

(e)To every linear transformation T of $\Re^k$ to itself corresponds a real number $\Delta(T)$ such that $m(T(E)) = \Delta(T)m(E)$ for all $E \in \mathfrak{M}$.

2.3. **Complex Measures.**

Theorem 40. *The total variation $|\mu|$ of a complex measure on $\mathfrak{M}$ is a positive bounded measure on $\mathfrak{M}$.*

Theorem 41. *Suppose μ, λ, λ_1, and λ_2 are measures on $\mathfrak{M}$ and μ is positive. Then*

(a) If λ is concentrated on A, so is $|\lambda|$.

(b) If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.

(c) If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.
(d) If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.
(e) If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.
(f) If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.
(g) If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.

Theorem 42. *Suppose μ and λ are measures on a σ-algebra $\mathfrak{M}$, μ is positive and λ is complex. Then the following two are equivalent:*
(a) $\lambda \ll \mu$; (b)To every $\epsilon > 0$ corresponds a $\delta > 0$ such that $|\lambda(E)| < \epsilon$ for all $E \in \mathfrak{M}$.

Theorem 43. *If μ is a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in X, then there is a function $\omega \in \mathcal{L}^1(\mu)$ such that $0 < \omega(x) < 1$ for all $x \in X$.*

Theorem 44. *The Theorem of Lebesgue-Radon-Nikodym*
Let μ be a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ on X and λ in a complex measure on $\mathfrak{M}$. Then
(a) There is a unique pair of complex measures λ_a and λ_s on $\mathfrak{M}$ such that $\lambda = \lambda_a + \lambda_s$ and $\lambda_a \ll \mu$, $\lambda_s \perp \mu$.
(b) There is a unique $h \in \mathcal{L}^1(\mu)$ such that $\lambda_a(E) = \int_E h d\mu$ for every $E \in \mathfrak{M}$.

Theorem 45. *Let μ be a complex measure on a σ-algebra $\mathfrak{M}$ in X. Then there is a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and $d\mu = hd|\mu|$.*

Theorem 46. *Suppose μ is a positive measure on $\mathfrak{M}$, $g \in \mathcal{L}^1(\mu)$, and $\lambda(E) = \int_E g d\mu$ for all $E \in \mathfrak{M}$. Then $|\lambda|(E) = \int_E |g| d\mu$.*

Theorem 47. *Let μ be a real measure on $\mathfrak{M}$. Then there exists sets A and B such that $A \cup B = X$, $A \cap B = \emptyset$ and $\mu^+(E) = \mu(A \cap E)$, $\mu^-(E) = -\mu(B \cap E)$ for all $E \in \mathfrak{M}$.*

3. Inequalities

Theorem 48. *Fatou's Lemma*
If $f_n : X \to [0, \infty]$ is measurable, then $\int_X (\underline{\lim} f_n) d\mu \leq \underline{\lim} \int_X f_n d\mu$.

Theorem 49. *Jensen's Inequality*
Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in a set Ω such that $\mu(\Omega) = 1$. If f is a real function in $\mathcal{L}^1(\mu)$ and ϕ is convex, then $\phi(\int_\Omega f d\mu) \leq \int_\Omega (\phi \circ f) d\mu$.

Corollary 3. *For $\{\alpha_i\}_{i=1}^n$ such that $\alpha_i > 0$ and $\sum_{i=1}^n \alpha_i = 1$,*
$y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \leq \alpha_1 y_1 + \alpha_2 y_2 \cdots + \alpha_n y_n$ if $y_i \geq 0$.

Theorem 50. *Hardy's Inequality*
Suppose $1 < p < \infty$,$f \in \mathcal{L}^p(0, \infty)$, and $F(x) = \frac{1}{x} \int_0^x f(t) dt$ $(0 < x < \infty)$. Then $||F||_p \leq \frac{p}{p-1} ||f||_p$.

Theorem 51. *Suppose* $1/p+1/q=1$*,where* $1 \le p \le \infty$*, for* $f, h \in \mathcal{L}^p(\mu), g \in \mathcal{L}^q(\mu)$*,* $||fg||_1 \le ||f||_p ||g||_q$*,* $||f+h||_p \le ||f||_p + ||h||_p$*.*

Theorem 52. *Let* $\{u_\alpha : \alpha \in A\}$ *be an orthonormal set in a Hilbert space* H*, then* $\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \le ||x||^2$ *holds for all* $x \in H$*, where* $\hat{x}$ *is the Fourier coefficients of* x*.*

4. Abstract Integration

Theorem 53. *Let* f *and* g *be nonnegative real measurable functions on* X*. Then*
(a) If $f \le g$*, then* $\int_E f d\mu \le \int_E g d\mu$*.*
(b) If $A \subset B$*, then* $\int_A f d\mu \le \int_B f d\mu$*.*
(c) $\int_E cf d\mu = c \int_E f d\mu$*.*
(d) If $f(x) = 0$ *for all* $x \in E$*, then* $\int_E f d\mu = 0$ *even if* $\mu(E) = \infty$*.*
(e) if $\mu(E) = 0$*, then* $\int_E f d\mu = 0$*, even if* $f(x) = 0$ *for all* $x \in E$*.*
(f) If $f \ge 0$*, then* $\int_E f d\mu = \int_X \mathcal{X}_E f d\mu$*.*

Theorem 54. *Lebesgue's Monotone Convergence Theorem*
Let $\{f_n\}$ *be a sequence of measurable functions on* X *such that*
(a) $0 \le f_1 \le f_2 \le \cdots \le \infty$*, and (b)* $f_n \to f$ *as* $n \to \infty$*.*
Then f *is measurable and* $\int_X f_n d\mu \to \int_X f d\mu$ *as* $n \to \infty$*.*

Corollary 4. *If* $f_n : X \to [0, \infty]$ *is measurable and* $f = \sum_{n=1}^{\infty} f_n$*,*
then $\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$*.*

Theorem 55. *Let* f *and* $g \in \mathcal{L}^1(\mu)$ *and* α *and* β *be scalars. Then*
(a)$\alpha f + \beta g \in \mathcal{L}^1(\mu)$*, and* $\int_X (\alpha f + \beta g) d\mu = \alpha \int_X f d\mu + \beta \int_X g d\mu$*.*
(b)$|\int_X f d\mu| \le \int_X |f| d\mu$*.*

Theorem 56. *Suppose* $f \in \mathcal{L}^1(\mu)$*,* $\epsilon > 0$*, then there exists a* $\delta > 0$ *such that* $\int_E |f| d\mu < \epsilon$ *whenever* $\mu(E) < \delta$*.*

Theorem 57. *Dominated Convergence Theorem*
Suppose $\{f_n\}$ *is a sequence of complex measurable functions on* X *such that* $f = \lim_{n \to \infty} f_n$*. If there exists a function* $g \in \mathcal{L}^1(\mu)$ *such that* $|f_n| \le g$ *for all* $n \in \mathcal{N}, x \in X$*, then* $f \in \mathcal{L}^1(\mu)$ *and* $\lim_{n \to \infty} \int_X |f_n - f| d\mu = 0$*, and* $\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu$*.*

Theorem 58. *Lieb's Lemma*
Suppose $f_n : X \to \mathbb{C}$ *converges to* $f \in \mathcal{L}^1(\mu)$*, then* $lim(\int_X |f_n| - \int_X |f| - \int_X |f_n - f|) = 0$*.*

Theorem 59. *Suppose* $\mu(X) < \infty$*,* $\{f_n\}$ *is a sequence of bounded complex measurable functions on* X*, and* $f_n \to f$ *uniformly on* X*. Then* $\lim \int_X f_n d\mu = \int_X f d\mu$*.*

Theorem 60. *Suppose* $\{f_n\}$ *is a sequence of complex measurable functions defined a.e. on* X *such that* $\sum_1^{\infty} \int_X |f_n| d\mu < \infty$*. Then the series* $f = \sum_1^{\infty} f_n$ *converges for almost all* x*,* $f \in \mathcal{L}^1(\mu)$*, and* $\int_X f d\mu = \sum_1^{\infty} \int_X f_n d\mu$*.*

Theorem 61. *(a)Suppose $f : X \to [0, \infty]$ is measurable, $E \in \mathfrak{M}$ and $\int_E f d\mu = 0$. Then $f = 0$ a.e. on E.*

(b)Suppose $f \in \mathcal{L}^1(\mu)$ and $\int_E f d\mu = 0$ for every $E \in \mathfrak{M}$. Then $f = 0$ a.e. on X.

(c) Suppose $f \in \mathcal{L}^1(\mu)$ and $|\int_X f d\mu| = \int_X |f| d\mu$. Then there is a constant α such that $\alpha f = |f|$ a.e. on X.

Theorem 62. *Suppose $\mu(X) < \infty$, $f \in \mathcal{L}^1(\mu)$, S is a closed set in the complex plane, and the avearges $A_E(f) = \frac{1}{\mu(E)} \int_E f d\mu$ lie in S for every $E \in \mathfrak{M}$ with $\mu(E) > 0$. Then $f(x) \in S$ for almost all $x \in X$.*

5. Approximation Theorems

Theorem 63. *Let $f : X \to [0, \infty]$ be measurable. There exists simple measurable functions s_n on X such that (a) $0 \leq s_1 \leq s_2 \leq \cdots \leq f$, (b) $s_n \to f(x)$ as $n \to \infty$, for every $x \in X$.*

Theorem 64. *Let $f : X \to [-\infty, +\infty]$ be measurable, then there exist $\{\phi_n\}_{n\in\mathbb{N}}$, a sequence of simple functions such that $|\phi_n| \leq |\phi_{n+1}|$ for all n and $\lim \phi_n = f$.*

Theorem 65. *Lusin's Theorem*

Let X be a locally compact Hausdorff space and μ be a positive measure as in Riesz Representation Theorem.

Suppose f is a complex measurable function on X, $\mu(\{f \neq 0\}) < \infty$ and $\epsilon > 0$. Then there exists a $g \in C_c(X)$ such that $\mu(\{f \neq g\}) < \epsilon$ and $\sup |g| \leq \sup |f|$.

Corollary 5. *Assume hypothesis in Lusin's Theorem and $|f| \leq 1$. Then there exists $\{g_n\} \in C_c(X)$ with $|g_n| \leq 1$ and $f = \lim g_n$ a.e.*

Theorem 66. *Egoroff's Theorem*

If $\mu(X) < \infty$ and $\{f_n\}$ is a sequence of complex measurable functions converging a.e. on X, then given $\epsilon > 0$, there exists $E \subset X$ with $\mu(X - E) < \epsilon$ such that $\{f_n\}$ converges uniformly on E.

Theorem 67. *Letμ be a positive measure on X and $\{f_n\}$ be a sequence of complex measurable functions. Then*

(a) If $f_n \to f$ a.e., then $f_n \to f$ in measure.

(b) If $1 \leq p \leq \infty$, $f_n \in \mathcal{L}^p(\mu)$ and $f_n \to f$ in $\mathcal{L}^p(\mu)$, then $f_n \to f$ in measure.

(c) If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence that converges to f a.e.

Theorem 68. *If $1 \leq p \leq \infty$ and $\{f_n\}$ is Cauchy in $\mathcal{L}^p(\mu)$ with limit f, then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to f.*

Theorem 69. *Let $1 \leq p < \infty$ and S be the class of complex measurable simple functions such that $\mu(\{s \neq 0\}) < \infty$. Then S is dense in $\mathcal{L}^p(\mu)$.*

Theorem 70. *Let X be a locally compact Hausdorff space. For $1 \leq p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$.*

Theorem 71. *Let X be a locally compact Hausdorff space, then $C_0(X)$ is the completion of $C_c(X)$ with respect to the norm $||f|| = \sup |f|$.*

Theorem 72. *The Vitali-Caratheodory Theorem*

Suppose $f \in \mathcal{L}^1(\mu)$ and is real valued. Given $\epsilon > 0$, then there exists functions u and v on X such that $u \leq f \leq v$, u is upper semicontinuos and bounded above, v is lower semicontinuos and bounded below, and $\int_X (v - u) d\mu < \epsilon$.

Theorem 73. *Let f be a real valued Lebesgue measurable function on $\Re^k$. Then there exist Borel functions g and h such that $g = h$ a.e. and $g \leq f \leq h$ on X.*

6. Examples of Vector Spaces

6.1. **Euclidean Spaces.** For $n \in \mathbb{N}$, define P_n be the set of all $x \in \Re^k$ whose coordinates are integral multiples of 2^{-n} and Ω_n be the collection of all 2^{-n}-boxes with corners at points of P_n, where a δ-box cornered at $a = \alpha_1, \alpha_2, \ldots, \alpha_k$, $Q(a;\delta)$ is defined to be $Q(a;\delta) = \{x : \alpha_i \leq \xi < \alpha_i + \delta, 1 \leq i \leq k\}$.

Theorem 74. *(a) If n is fixed, each $x \in \Re^k$ lies in one and only one member of Ω_n.*

(b) If $Q' \in \Omega_n$, $Q'' \in \Omega r$, and $r < n$, then either $Q' \subset Q''$ or $Q' \cap Q'' = \emptyset$.

(c) If $Q \in \Omega_r$, then $vol(Q) = 2^{-rk}$; and if $n > r$, the set P_n has exactly $2^{k(n-r)}$ points in Q.

(d) Every nonempty open set in $\Re^k$ is a countable union of disjoint boxes belonging to $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \cdots$.

6.2. $\mathcal{L}^p(\mu)$ **Spaces.**

Theorem 75. *For $1 \leq p \leq \infty$ and every positive measure μ, $\mathcal{L}^p(\mu)$ is a complete metric space.*

Theorem 76. *If $r < p < s$, then $\mathcal{L}^r(\mu) \cap \mathcal{L}^s(\mu) \subset \mathcal{L}^p(\mu)$.*

Theorem 77. *If $\mu(X) < \infty$, then $\mathcal{L}^r(\mu) \subset \mathcal{L}^s(\mu)$ whenever $r < s$.*

Theorem 78. *Suppose p and q are conjugate exponents, $1 \leq p < \infty$, μ is a σ-finite positive measure on X, then $\mathcal{L}^q(\mu)$ is isometrically isomorphic to the dual space of $\mathcal{L}^p(\mu)$.*

Theorem 79. *Suppose p and q are conjugate exponents, $1 < p < \infty$, μ is a positive measure on X, then $\mathcal{L}^q(\mu)$ is isometrically isomorphic to the dual space of $\mathcal{L}^p(\mu)$.*

Theorem 80. *Suppose $1 \leq p < \infty$, then $\mathcal{L}^p(\Re^k)$ is separable. $\mathcal{L}^\infty(\Re^k)$ is not separable.*

6.3. **Hilbert Spaces, $\mathcal{H}$.**

Theorem 81. *If M is a subspace of $\mathcal{H}$, so is $\overline{M}$.*

Theorem 82. *Let $x \in \mathcal{H}$ and M is a subspace of $\mathcal{H}$, then $x^{\perp}$ and $M^{\perp}$ are closed subspace of $\mathcal{H}$.*

Theorem 83. *Every nonempty, closed, convex set E in a Hilbert space $\mathcal{H}$ contains a unique element of smallest norm.*

Theorem 84. *Let M be a closed subspace of $\mathcal{H}$ and $x \in \mathcal{H}$ has a unique decomposition $x = Px + Qx$, where $Px \in M$ and $Qx \in M^{\perp}$. And*
(a) Px and Qx are the nearest points to x in M and in $M^{\perp}$, respectively.
(b) $||x||^2 = ||Px||^2 + ||Qx||^2$.
(c) The mappings P and Q are linear.

Corollary 6. *If M is a closed subspace of $\mathcal{H}$ and $M \neq \mathcal{H}$, then there exists $x \neq 0$ in $M^{\perp}$.*

Theorem 85. *If L is a continuos linear functional on $\mathcal{H}$, then there is a unique $y \in \mathcal{H}$ such that $Lx = (x, y)$ for all $x \in \mathcal{H}$.*

Theorem 86. *Suppose that $\{u_\alpha : \alpha \in A\}$ is an orthonormal set in $\mathcal{H}$ and that F is a finite subset of A. Let M_F be the span of $\{u_\alpha : \alpha \in A\}$.*
(a) If ϕ is a complex function on A that vanishes outside F, then there is an $y \in M_F$, namely $y = \sum_{\alpha\in F} \phi(\alpha)u_\alpha$ such that $\hat{y}(\alpha) = \phi(\alpha)$ for every $\alpha \in A$.
(b) If $x \in \mathcal{H}$ and $s_F(x) = \sum_{\alpha\in F} \hat{x}(\alpha)u_\alpha$, then $||x - s_F(x)|| < ||x - s||$ for every $s \in M_F$ except for $s = s_F(x)$.

Theorem 87. *Assuming Hausdorff Maximality Theorem, then $\mathcal{H}$ has a maximal orthonormal system $\{u_\alpha : \alpha \in A\}$ such that*
(a) The set P of all finite linear combinations of members of $\{u_\alpha\}$ is dense in $\mathcal{H}$.
(b) $\sum_{\alpha\in A} |\hat{x}(\alpha)|^2 = ||x||W^2$ for all $x \in \mathcal{H}$.
(c) $\sum_{\alpha\in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)} = (x, y)$ for all $x, y \in \mathcal{H}$.

Corollary 7. *$\mathcal{H}$ is isometrically isomorphic to some $l^2(A)$.*

Theorem 88. *$\mathcal{H}$ is separable if and only if it contains a maximal orthonormal system that is countable.*

Theorem 89. *The Hilbert cube is compact.*

6.4. **Banach Spaces.** Refer to relevant theorems in the section *Functionals and Linear Transformations.*

7. Functionals and Linear Transformations

Theorem 90. *For a linear transformation Λ from a normed linear space X into a normed linear space Y, each of the following three conditions implies the other two:*
(a) Λ is bounded;
(b) Λ is continuos;
(c) Λ is continuos at one point of X.

Theorem 91. *The Banach-Steinhaus Theorem*
Suppose X is a Banach space, Y is a normed linear space, and $\{\Lambda_\alpha\}_{\alpha\in A}$ is a collection of bounded linear transformations of X into Y. Then either there exists $M < \infty$ such that $||\Lambda_\alpha|| \leq M$ for all $\alpha \in A$, or $\sup_{\alpha\in A} ||\Lambda_\alpha x|| = \infty$ for all x in some dense $G_\delta \subset X$.

Theorem 92. *The Open Mapping Theorem*
Let U and V be the open unit balls of Banach spaces X and Y. To every bounded linear transformation Λ of X onto Y there corresponds a $\delta > 0$ such that $\Lambda(U) \supset \delta V$.

Corollary 8. *Assume the hypothesis in the last theorem and Λ is one-to-one, then there exists a $\epsilon > 0$ such that $||\Lambda x|| \geq \epsilon||x||$ for all $x \in X$. That is, Λ^{-1} is also a bounded linear transformation.*

Theorem 93. *The Hahn-Banach Theorem*
If M is the subspace of a normed linear space X and if f is a bounded linear functional on M, then f can be extended to a bounded linear functional F on X such that $||F|| = ||f||$.

Corollary 9. *Assume the hypothesis in the last theorem and let $x_0 \in X$.Then x_0 is in $\overline{M}$ if and only if there is no bounded linear functional f on X such that $f(x) = 0$ for all $x \in M$ but $f(x_0) \neq 0$.*

Corollary 10. *If X is a normed linear space and $x_0 \neq 0, x_0 \in X$, there is a bounded linear functional f on X of norm 1 such that $f(x_0) = ||x_0||$.*

Theorem 94. *The Reisz Representation Theorem for positive linear functionals*
Let X be a locally compact Hausdorff space and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathfrak{M}$ on X that contains all Borel sets in X, and there is a unique positive measure μ on $\mathfrak{M}$ that represents Λ in the sense that $\Lambda f = \int_X f d\mu$ for all $f \in C_c(X)$, and with the following properties:
(a) $\mu(K) < \infty$ for every compact set $K \subset X$.
(b) For every $E \in \mathfrak{M}$, we have $\mu(E) = \inf \mu(V) : E \subset V, V open$.
(c) $\mu(E) = \sup \mu(K) : K \subset E, K compact$ for all open set E and every E with $\mu(E) < \infty$.
(d) $\mathfrak{M}$ is complete.

Theorem 95. *The Reisz Representation Theorem for bounded linear functionals*

If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, in the sense that $\Phi f = \int_X f d\mu$ for all $f \in C_0(X)$ with norm $||\Phi|| = |\mu|(X)$.